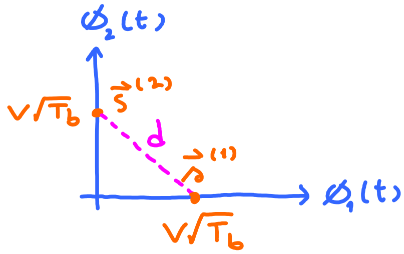
Q1 Binary Orthogonal Scheme

Wednesday, September 11, 2013
10:07 AM

(a)

From HW 1, we have already used GSOP to derive the constellation :



Remark: we know that, for equiprobable binary signalling schemes,

$$P(\mathcal{E}) = Q\left(\frac{d}{2\sigma}\right).$$

Pythagorus' theorem

The distance between the two points is $d = \sqrt{2}\, V\sqrt{T_b}$.
So, the corresponding probability of (decoding) error is

$$P(\mathcal{E}) = Q\left(\frac{d}{2\sigma}\right) = Q\left(\frac{V\sqrt{2T_b}}{2\sigma}\right) = Q\left(\frac{V\sqrt{T_b}}{\sqrt{2}\,\sigma}\right)$$

For AWGN channel with PSD $\frac{N_0}{2}$, we have $\sigma^2 = \frac{N_0}{2}$.

Therefore, $P(\mathcal{E}) = Q\left(\frac{V\sqrt{T_b}}{\sqrt{N_0}}\right) = Q\left(V\sqrt{\frac{T_b}{N_0}}\right)$.

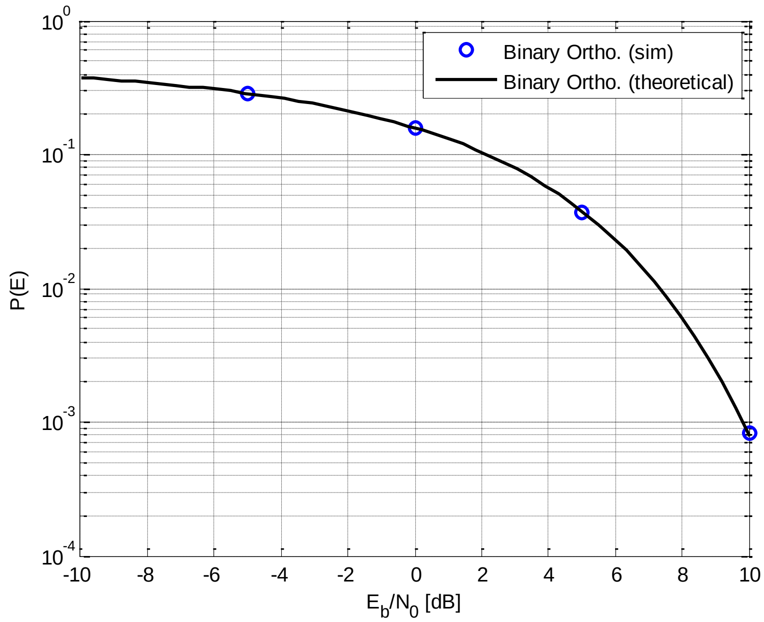
(b) The energy of both points is $V^2 T_b$. So, $E_s = V^2 T_b$.

Here, $M = 2$. So, $E_b = \frac{E_s}{\log_2 2} = V^2 T_b$.

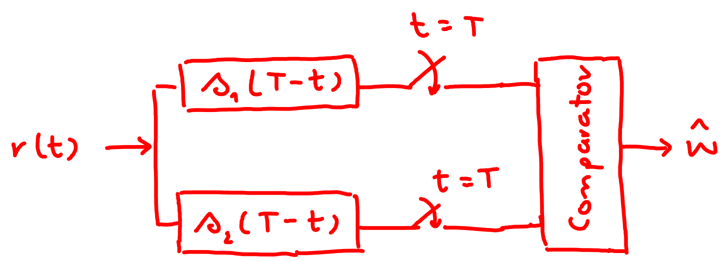
Rewriting $P(\mathcal{E})$ in part (a), we have

$$P(\mathcal{E}) = Q\left(\sqrt{\frac{V^2 T_b}{N_0}}\right) = \underbrace{Q\left(\sqrt{\frac{E_b}{N_0}}\right)}.$$

In fact, we can answer this directly by realizing that this is an equiprobable binary orthogonal signaling scheme.



(c) The optimal detector can be implemented from matched filter as followed

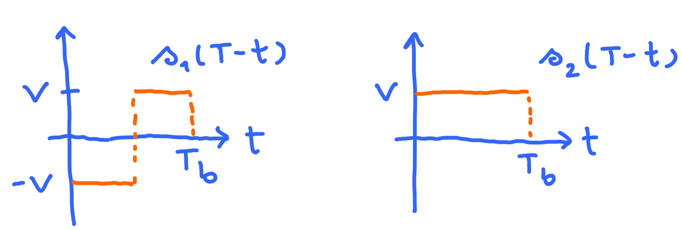


Note that

1. There is no need for the bias term because the waveforms are equally likely and of equal energy.

2. To make matched filter causal, we choose $T \geq T_b$.
   To minimize delay, we choose $T = T_b$.
   In which case, the plots for $s_1(T-t)$ and $s_2(T-t)$ are shown below:

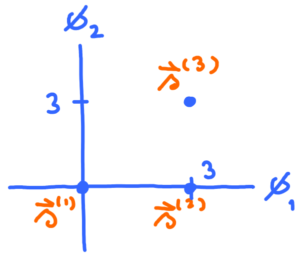
   $(s_i(T-t) = s_i(-(t-T)).)$



Q2 Minimum Energy for Constellation

Thursday, September 05, 2013
3:31 PM

(a)



$$\vec{s}^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \vec{s}^{(2)} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \qquad \vec{s}^{(3)} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$E_s = \frac{1}{3}\left(0^2 + 3^2 + 3^2 + 3^2\right) = 9$$

(b)
$$E_s = \sum_{i=1}^{3} p_i \sum_{j=1}^{2} \left(\vec{s}_j^{(i)}\right)^2 = \frac{1}{3}\left(\sum_{i=1}^{3}\left(\vec{s}_1^{(i)}\right)^2 + \sum_{i=1}^{3}\left(\vec{s}_2^{(i)}\right)^2 \right)$$

$$= \frac{1}{3}\left((0-a_1)^2 + (2-a_1)^2 + (2-a_1)^2 \right) + \frac{1}{3}\left((0-a_2)^2 + (0-a_2)^2 + (2-a_2^2) \right)$$

In general, we have to minimize terms of the form

$$\sum_i p_i (x_i - a)^2 = \mathbb{E}\left[(X-a)^2\right] = \mathbb{E}\left[(X - \underbrace{\mathbb{E}X + \mathbb{E}X}_{0} - a)^2 \right]$$

$$= \operatorname{Var} X + 2\, \underbrace{\mathbb{E}\left[(X - \mathbb{E}X)\right]}_{0} (\mathbb{E}X - a) + (\mathbb{E}X - a)^2$$

$$= \operatorname{Var} X + \underbrace{(\mathbb{E}X - a)^2}$$

the only term that depends on a.
Minimum value of 0 is achieved
when $a = \mathbb{E}X$.

So, the minimum value occurs when $a = \mathbb{E}X = \sum_i p_i x_i$

Minimum E_s occurs when

$$a_1 = \frac{1}{3}(0 + 3 + 3) = 2. \qquad a_2 = \frac{1}{3}(0 + 0 + 3) = 1.$$

Q3 24-QAM

Wednesday, September 11, 2013
10:23 AM

1. For 1-D standard rectangular M-PAM

$$P(\mathcal{E}\mid \vec{S}=\vec{s}^{(i)}) = \begin{cases} Q\left(\frac{d}{2\sigma}\right), & i = 1, M \\ 2Q\left(\frac{d}{2\sigma}\right), & i = 2,3,\dots,M-1 \end{cases}$$

$$\text{So, } P(\mathcal{E}) = \frac{1}{M}\left(2\times q + (M-2)2q\right) = \frac{1}{M}\left((M-1)2q\right) = 2\frac{M-1}{M}q = 2\frac{M-1}{M}Q\left(\frac{d}{2\sigma}\right)$$

$$A = 2\frac{M-1}{M} \overset{M=24}{=} \frac{23}{12}$$

$$B = 0$$

For 2-D standard rectangular M-QAM

To find $P(\mathcal{E})$, recall that we have three cases of points

Case	#	$P(\mathcal{E}\mid \vec{S}=\vec{s}^{(i)})$
corner	n_1	$2q-q^2$
middle	n_2	$3q-2q^2$
center	n_3	$4q-4q^2$

$$q = Q\left(\frac{d}{2\sigma}\right)$$

Therefore, we simply count the # points in each case and use those numbers as weights for $P(\mathcal{E})$:

$$P(\mathcal{E}) = \frac{1}{M}\left(n_1(2q-q^2) + n_2(3q-2q^2) + n_3(4q-4q^2)\right)$$

$$\Rightarrow A = \frac{1}{M}\left(2n_1 + 3n_2 + 4n_3\right)$$

$$\Rightarrow B = -\frac{1}{M}\left(n_1 + 2n_2 + 4n_3\right)$$

$M = M_1 \times M_2$	$n_1 = 4$	$n_2 = 2(M_1-2)+2(M_2-2)$	$n_3 = M-n_1-n_2$	A	B
2×12	4	20	0	$68/24 = 17/6$	$-44/24 = 11/6$
3×8	4	14	6	$74/24 = 37/12$	$-56/24 = 7/3$
4×6	4	12	8	$76/24 = 19/6$	$-60/24 = 5/2$

2. To change $\frac{d}{2\sigma}$ to $\frac{E_b}{N_0}$, we need to find E_b.

To do this, we start with E_s ← average energy per symbol.

Remark: There are many ways to find E_s. As long as you can find the coordinates of the points in the constellation, then, it is straightforward to find the energy of each point and then average all the energy. This can be done easily in MATLAB. However, here, we show how to derive the answer analytically.

Standard 1-D M-PAM:

Start with $(1, 2, 3, \dots, M)\times d$

↑ this changes the spacing btw the pts to d.

Then, we shift the constellation so that the center is at origin. To do this, we simply subtract the average out.

at origin. To do this, we simply subtract the average out.

$$(1,2,3,\dots,M)\times d \;-\; \underbrace{\frac{1}{M}\left(\sum_{k=1}^{M} k\right)d}_{\text{call this as } m \,=\, \frac{1}{\cancel{M}}\frac{(M+1)\cancel{M}}{2}d}$$

Viewing this as a RV, we may think about

RV $U \sim$ uniform on $1,2,\dots,M$

RV $S = dU - \mathbb{E}[dU] = d(U - \mathbb{E}U)$

So, $\mathbb{E}S = 0$ and $\mathbb{E}[S^2] = \operatorname{Var} S = d^2 \operatorname{Var} U$

Next, we find the average energy:

$$E_s = \frac{1}{M}\left(\sum_{k=1}^{M}(kd-m)^2\right) = \frac{1}{M}\left(\sum_{k=1}^{M}k^2d^2 - \underbrace{2md\overbrace{\sum_{k=1}^{M}k}^{M\times m}}_{-2Mm^2} + m^2M\right)$$

$$= \frac{1}{M}\left(\sum_{k=1}^{M}k^2d^2 - Mm^2\right)$$

$$= d^2\left(\left(\frac{1}{M}\sum_{k=1}^{M}k^2\right) - \left(\frac{1}{M}\sum_{k=1}^{M}k\right)^2\right) = d^2\left(\frac{1}{3}(M^2-1) - \left(\frac{M+1}{2}\right)^2\right)$$

$$= d^2\left((M+1)\left(\frac{M-1}{3} - \frac{M+1}{4}\right)\right) = d^2\,\frac{(M+1)(M-1)}{12} = \frac{1}{12}d^2(M^2-1)$$

Some facts about summation:

$$\sum_{k=1}^{M} k = \frac{M(M+1)}{2}$$

$$\begin{aligned} A &= 1 + 2 + \cdots + M \\ A &= M + M-1 + \cdots + 1 \quad + \\ \hline 2A &= \underbrace{(M+1) + (M+1) + \cdots + (M+1)}_{M \text{ terms}} \end{aligned}$$

$$\sum_{k=1}^{M} k^2 = \sum_{k=1}^{M} k(k-1) + \sum_{k=1}^{M} k$$

$$\sum_{k=1}^{M} k(k-1) = 1\cdot 0 + 2\cdot 1 + 3\cdot 2 + \cdots + M(M-1)$$

$$= 1\cdot 0\cdot 3 + 2\cdot 1\cdot 3 + 3\cdot 2\cdot 3 + \cdots + M(M-1)\cdot 3$$

$$= 1\cdot 0\cdot\frac{2-(-1)}{3} + 2\cdot 1\cdot\frac{3-0}{3} + \cdots + M(M-1)\frac{(M+1)-(M-2)}{3}$$

$$= \frac{1}{3}\Big((\cancel{2\cdot 1\cdot 0}) - (1\cdot 0\cdot(-1)) + (\cancel{3\cdot 2\cdot 1}) - (\cancel{2\cdot 1\cdot 0}) + \cdots + (M+1)(M)(M-1) - \cancel{M(M-1)(M-2)}\Big)$$

$$= \frac{1}{3}M(M+1)(M-1)$$

$$\sum_{k=1}^{M} k^2 = \frac{1}{3}M(M+1)(M-1) + \frac{M(M+1)}{2} = M(M+1)\left(\frac{M-1}{3} + \frac{1}{2}\right)$$

$$= \frac{1}{6}M(M+1)(2M+1)$$

Alternatively, we have $E_s = \mathbb{E}[S^2] = d^2 \operatorname{Var} U = d^2\left(\left(\frac{1}{M}\sum_{k=1}^{M}k^2\right) - \left(\frac{1}{M}\sum_{k=1}^{M}k\right)^2\right)$ ← same as above.

$$= d^2\left(\frac{M^2-1}{12}\right)$$

Knowing E_s, we can then find $E_b = E_s/\log_2 M = \frac{1}{12}\,\frac{d^2(M^2-1)}{\log_2 M}$

$$d^2 = \frac{12(\log_2 M)\,E_b}{}$$

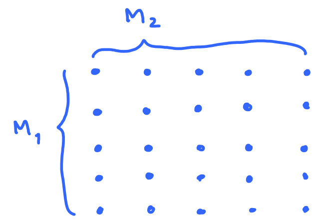
M^2-1

$$\frac{d}{2\sigma} = \sqrt{\frac{d^2}{2N_0}} = \sqrt{\underbrace{\frac{6}{M^2-1}(\log_2 M)}\frac{E_b}{N_0}}$$

$$C = \frac{6}{M^2-1}(\log_2 M) \overset{M=24}{=} \frac{6}{575}\log_2 24$$

Now for 2-D standard M-ary QAM.

Suppose $M = M_1 \times M_2$.



$$E_s = \mathbb{E}\left[\|\vec{s}\|^2\right] = \mathbb{E}\left[(s_1)^2 + (s_2)^2\right] = \mathbb{E}\left[s_1^2\right] + \mathbb{E}\left[s_2^2\right]$$

the first component of $\vec{s}$

the second component of $\vec{s}$

Let $U_j \sim$ Uniform on $1:M_j$.

Define $S_j = dU_j - \mathbb{E}[dU_j] = d(U_j - \mathbb{E}U_j)$

Then, $\mathbb{E}\left[s_1^2\right] = d^2 \operatorname{Var} U_j = d^2 \frac{M_j^2-1}{12}$

Therefore, $E_s = \frac{d^2}{12}\left(M_1^2 + M_2^2 - 2\right)$

$$E_b = \frac{d^2}{12}\frac{M_1^2+M_2^2-2}{\log_2 M} \Rightarrow d^2 = \frac{12}{M_1^2+M_2^2-2}\log_2 M \; E_b$$

$$\frac{d}{2\sigma} = \sqrt{\frac{d^2}{4\sigma^2}} = \sqrt{\frac{d^2}{2N_0}} = \sqrt{\underbrace{\frac{6}{M_1^2+M_2^2-2}(\log_2 M)}\frac{E_b}{N_0}}$$

$$C = \frac{6}{M_1^2+M_2^2-2}\log_2 M$$

(i) Summary:

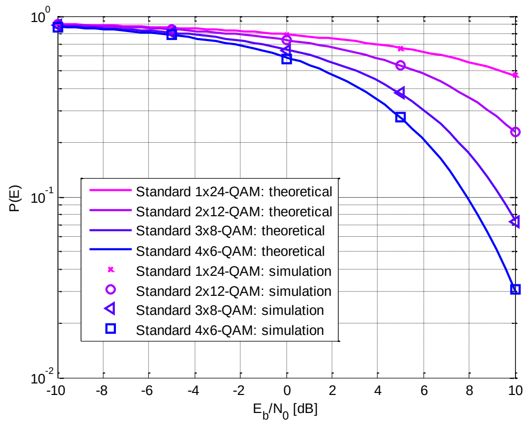
$M = M_1 \times M_2$	A	B	C
1×24	$\frac{23}{12}$	0	$\frac{6}{575}\times\log_2 24$
2×12	$\frac{17}{6}$	$-\frac{11}{6}$	$\frac{3}{73}\times\log_2 24$
3×8	$\frac{37}{12}$	$-\frac{7}{3}$	$\frac{6}{71}\times\log_2 24$
4×6	$\frac{19}{6}$	$-\frac{5}{2}$	$\frac{3}{25}\times\log_2 24$

(ii) For small $\frac{E_b}{N_0}$ (when $\frac{E_b}{N_0}\to 0$),

$$q = Q\left(\sqrt{c\frac{E_b}{N_0}}\right) \to Q(0) = 0.5$$

$$P(\mathcal{E}) \to \frac{A}{2} + \frac{B}{4} = \frac{23}{24} \approx 0.9583 \quad \text{(same for all constellation)}$$

For larger $\frac{E_b}{N_0}$, the plots of $P(\mathcal{E})$ shows that the performance is better when the constellation is closer to being a square.



Intuitively, we note that $P(\mathcal{E})$ depends strongly on the distances btw the points in the constellation. The "square" constellation uses less average energy because the points are closer to the origin. So, for a given average energy, the "square" constellation enjoys greater distances btw its point and hence better $P(\mathcal{E})$.

(iii) See the plots in part (ii).